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ON SYMMETRIC FUNCTIONS.

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The object of the following paper is to call attention to certain methods of treating symmetric functions of the roots of an equation from two quite different standpoints. From the first standpoint the symmetric function as a whole as expressed in terms of the coefficients of the equation is considered; from the second, the isolation of one of these terms with its numerical coefficient is the object of investigation. Accordingly the paper is divided into two chief divisions. We proceed to the first.

I. SYMMETRIC FUNCTIONS AS A WHOLE.

It is not the writer's purpose to reproduce the many formulas and methods which are already given in works on algebra for expressing symmetric functions as a whole. Only one of them will be noticed, and attention called to two others, as follows:

A. FORMULAS.

1. BRIOSCHI'S FORMULA IN TERMS OF THE s'S.

(1). *Statement of the Formula.*

In 1854 in the *Annali di Tortolini* (t. 5, pp. 427-8), Brioschi gave without proof the following formula:

$$\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\ldots\ldots\alpha_n^{p_n}=\begin{vmatrix} u_{11} & u_{12}\ldots\ldots u_{1n} \\ u_{21} & u_{22}\ldots\ldots u_{2n} \\ \ldots\ldots & \ldots\ldots\ldots\ldots \\ u_{n1} & u_{n2}\;\ldots\ldots\; u_{nn} \end{vmatrix}$$

where $u_{11}=s_{p_1}$, $u_{22}=s_{p_2}$, $u_{12}u_{21}=s_{p_1+p_2}$, $\ldots\ldots u_{rs}u_{st}u_{tu}\ldots\ldots u_{xr}=s_{p_r+p_s+p_t\ldots\ldots p_x}$.

(2). *An Inadequate Statement.*

This formula seems to have lost the clearness of Brioschi's statement in Faà di Bruno's *Binäre Forman*, p. 8, so that the statement there made is inadequate. The writer has given a correction and proof of the formula (June-July number, 1898, pp. 161-4, of the MONTHLY).

(3). *Critical Value of the Formula.*

Taken in connection with the formula

$$s_r=\frac{(-1)^r}{a_0{}^r}\begin{vmatrix} a_1 & a_0 & 0 & 0\ldots\ldots\ldots\ldots 0 \\ 2a_2 & a_1 & a_0 & 0\ldots\ldots\ldots\ldots 0 \\ 3a_3 & a_2 & a_1 & a_0\quad 0\ldots\ldots\ldots 0 \\ \ldots\ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots a_0 \\ ra_r & a_{r-1} & a_{r-2} & \ldots\ldots\ldots\ldots a_1 \end{vmatrix}$$

of degree r and weight r as seen by developing in terms of the elements of the last line, it affords an example of a complete theoretical solution of the problem of expressing the symmetric function $\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\ldots\ldots\alpha_n^{p_n}$ in terms of the coefficients of the equation, and gives at once the theorems concerning rationality and weight; but on account of the theorem concerning order it is clear that this expression must contain many superfluous terms which destroy in the working out, and is therefore little adapted to practical purposes.

2. GORDAN'S FORMULA IN TERMS OF THE a'S.

(1). *Statement of the Formula.*

The following theorem is due to Professor Gordan, and was stated to the writer by him. If we multiply the alternating function

$$D=\begin{vmatrix} 1 & \alpha_1 & \alpha_1^2\ldots\ldots\alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2\ldots\ldots\alpha_2^{n-1} \\ \ldots & \ldots & \ldots\ldots\ldots\ldots \\ 1 & \alpha_n & \alpha_n^2\ldots\ldots\alpha_n^{n-1} \end{vmatrix}$$

by $\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\ldots\ldots\alpha_n^{p_n}$ supposing the p's to be all different, [If k_1 of the p's are p_1, k_2, p_2, $\ldots\ldots k_r$, p_r, we must divide the result by $k_1!\,k_2!\,\ldots\ldots k_r!$ to obtain Σ.] the result is the same as if we apply the n exponents of Σ to the columns of D in all $n!$ permutations of the same, and take the sum of the $n!$ determinants so resulting. This operation Professor Gordan indicates as $[p_1p_2\ldots\ldots p_n]$ and its

expansion as $\Sigma(p_{i_1} p_{i_2}+1 \ldots \ldots p_{i_n}+n-1)$, where $i_1, i_2 \ldots\ldots i_n$ form a permutation of the numbers 1, 2, ...n. Thus we have the formula

$$D\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\ldots\ldots\alpha_n^{p_n}=[p_1 p_2 \ldots\ldots p_n]=\Sigma(p_{i_1}p_{i_2}+1\ldots\ldots p_{i_n}+n-1).$$

(2). *Application of the Formula.*

To apply it we notice that the matrices

$$\left|\begin{matrix} 1 & \alpha_1 & \alpha_1^2\ldots\ldots\alpha_1^{2n-1} \\ 1 & \alpha_2 & \alpha_2^2\ldots\ldots\alpha_2^{2n-1} \\ \ldots & \ldots & \ldots \\ 1 & \alpha_n & \alpha_n^2\ldots\ldots\alpha_n^{2n-1} \end{matrix}\right| \text{ and } \left|\begin{matrix} a_n & a_{n-1} & a_{n-2}\ldots\ldots a_0 & 0 & 0\ldots\ldots 0 \\ 0 & a_n & a_{n-1}\ldots\ldots a_1 & a_0 & 0\ldots\ldots 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0\ldots 0 & a_n\ldots\ldots & \ldots\ldots & \ldots a_1 a_0 \end{matrix}\right|$$

correspond, and from the theorem that "the corresponding determinants of corresponding matrices are proportional,"* that

$$D=\rho\left|\begin{matrix} a_0 & 0 & 0\ldots\ldots 0 \\ a_1 & a_0 & 0\ldots\ldots 0 \\ \ldots & \ldots & \ldots \\ a_{n-1} & \ldots\ldots & a_1 a_0 \end{matrix}\right|=\rho a_0^n.$$

When, as often happens, any of the numbers p_r+i_r, p_s+i_s, in a parenthesis, are equal, such symbolic parenthesis reduces to zero, because the determinant thereby signified has at least two columns equal. The values of the non-vanishing symbols are then read off from the second matrix by the help of the theorem concerning corresponding matrices, the factor ρ being added and the correct sign factor of the corresponding determinant. The factor ρ will then divide out and the result gives

$$a_0^n\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\ldots\ldots\alpha_n^{p_n}$$

which equals the algebraic sum of a number of determinants in the a's of the nth order.

(3). *Proof of the Formula.*

I have proved this theorem as follows: We will begin by considering some simple examples.

a. It is required to find $\Sigma\alpha_1^{p_1}\alpha_2^{p_2}$ for a quadratic equation $a_0x^2+a_1x+a_2=0$. The matrices are

$$\left|\begin{matrix} 1 & \alpha_1 & \alpha_1^2 & \alpha_1^3 \\ 1 & \alpha_2 & \alpha_2^2 & \alpha_2^3 \end{matrix}\right| \text{ and } \left|\begin{matrix} a_2 & a_1 & a_0 & 0 \\ 0 & a_2 & a_1 & a_0 \end{matrix}\right|.$$

$$D=\left|\begin{matrix} 1 & \alpha_1 \\ 1 & \alpha_2 \end{matrix}\right|=\rho\left|\begin{matrix} a_0 & 0 \\ a_1 & a_0 \end{matrix}\right|=\rho a_0^2.$$

*Gordan, Determinants, p. 95.

We indicate $D\Sigma\alpha_1^{p_1}\alpha_2^{p_2}=(p_1p_2)$. We perform the operation indicated on the left hand side in detail. We have

$$(\alpha_2-\alpha_1)(\alpha_1^{p_1}\alpha_2^{p_2}+\alpha_1^{p_2}\alpha_2^{p_1})=\alpha_1^{p}\alpha_2^{p_2+1}+\alpha_1^{p_2}\alpha_2^{p_1+1}-\alpha_1^{p_1+1}\alpha_2^{p_2}-\alpha_1^{p_2+1}\alpha_2^{p_1}$$

$$=\begin{vmatrix}\alpha_1^{p_1}\alpha_1^{p_2+1}\\ \alpha_2^{p_1}\alpha_2^{p_2+1}\end{vmatrix}+\begin{vmatrix}\alpha_1^{p_2}\alpha_1^{p_1+1}\\ \alpha_2^{p_2}\alpha_2^{p_1+1}\end{vmatrix}$$

by taking the first and fourth, and the second and third terms together. If we had multiplied

$$\begin{vmatrix}1 & \alpha_1\\ 1 & \alpha_2\end{vmatrix} \text{ by } \Sigma\alpha_1^{p_1}\alpha_2^{p_2} \text{ as } \begin{vmatrix}1 & \alpha_1\\ 1 & \alpha_2\end{vmatrix}\alpha_1^{p_1}\,\alpha_2^{p_2}+\begin{vmatrix}1 & \alpha_1\\ 1 & \alpha_2\end{vmatrix}\alpha_1^{p_2}\alpha_2^{p_1},$$

we should have obtained

$$\begin{vmatrix}\alpha_1^{p_1}\alpha_1^{p_1+1}\\ \alpha_2^{p_2}\alpha_2^{p_2+1}\end{vmatrix}+\begin{vmatrix}\alpha_1^{p_2}\alpha^{p_2+1}\\ \alpha_2^{p_1}\alpha^{p_1+1}\end{vmatrix}.$$

If we compare with the previous result, we see that we could convert one into the other by exchanging in each the two secondary diagonal terms which is permitted. We choose the result first obtained and symbolize it by $(p_1p_2+1)+(p_2p_1+1)$, where in (p_1p_2+1) p_1 is the exponent of the first, and p_2+1 that of the second column of the determinant for which it stands. We write completely

$$D\Sigma\alpha_1^{p_1}\alpha_2^{p_2}=(p_1p_2)=(p_1p_2+1)+(p_2p_1+1).$$

b. We will apply this formula to the calculation of the symmetric functions required by the resultant of two quadratic forms,

$$fx=a_0x^2+a_1x+a_2=a_0(x-\alpha_1)(x-\alpha_2).$$
$$\phi x=b_0x^2+b_1x+b_2=b_0(x-\beta_1)(x-\beta_2).$$

The resultant of these forms is

$$a_0^2(b_0\alpha_1^2+b_1\alpha_1+b_2)(b_0\alpha_2^2+b_1\alpha_2+b_2)=$$
$$a_0^2(b_0^2\Sigma\alpha_1^2\alpha_2^2+b_0b_1\Sigma\alpha_1^2\alpha_2+b_0b_2\Sigma\alpha_1^2+b_1^2\Sigma\alpha_1\alpha_2+b_1b_2\Sigma\alpha_1+b_2^2).$$

c. We comprehend the results together in the following table. The sign of the determinant which corresponds to $(\alpha_1^{\lambda}\beta_1^{\mu})$ is by the theorem of corresponding matrices,

$$(-1)^{(1+2)+(\lambda+1+\mu+1)}=(-1)^{\lambda+\mu+1}.$$

We also observe that $(\lambda\lambda)=0$, and do not write 0.

Function	$[p_1 p_2]$	$\begin{matrix}(p_1 & p_2+1)\\(p_2 & p_1+1)\end{matrix}$	Determinants	Sign	Result
$2a_0^2\Sigma\alpha_1^2\alpha_2^2$	[2 2]	2(2 3)	$2\begin{vmatrix}a_2 & a_1\\0 & a_2\end{vmatrix}$	$(-1)^6=1$	$2a_2^2$
$a_0^2\Sigma\alpha_1^2\alpha_2$	[2 1]	(2 2) (1 3)	$\begin{vmatrix}a_2 & a_0\\0 & a_1\end{vmatrix}$	$(-1)^5=-1$	$-a_1a_2$
$a_0^2\Sigma\alpha_1^2$	[2 0]	(2 1) (0 3)	$-\begin{vmatrix}a_2 & 0\\0 & a_0\end{vmatrix}$ $\begin{vmatrix}a_1 & a_0\\a_2 & a_1\end{vmatrix}$	$-(-1)^4=-1$ $(-1)^4=1$	$-2a_0a_2+a_1^2$
$2a_0^2\Sigma\alpha_1\alpha_2$	[1 1]	2(1 2)	$2\begin{vmatrix}a_2 & 0\\0 & a_0\end{vmatrix}$	$(-1)^4=1$	$2a_2a_0$
$a_0^2\Sigma\alpha_1$	[1 0]	(1 1) (0 2)	$\begin{vmatrix}a_1 & 0\\a_2 & a_0\end{vmatrix}$	$(-1)^3=-1$	$-a_0a_1$

d. Substituting these values we have for the final result the function of the coefficients of the two quadratics

$$b_0^2a_2^2-b_0b_1a_1a_2+b_0b_2(a_1^2-2a_0a_2)+b_1^2a_0a_2-b_1b_2a_0a_1+b_2^2a_0^2.$$

e. We will now take another step and consider $\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}$. We wish to show that we obtain the same result when we apply the exponents in all possible permutations to the columns of D that we obtain when we apply them to the rows, and obtain $D\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}$. To prove this we need only show that to every term of the second formation corresponds exactly the same term in the first formation. First, it is clear that the number of terms in each formation is $(3\,!)^2=36$. Next, take a term like $\alpha_1^{p_3+1}\alpha_2^{p_1+2}\alpha_3^{p_2}$ of the second formation. It must have come from

$$\begin{vmatrix}\alpha_1^0 & \alpha_1 & \alpha_1^2\\ \alpha_2^0 & \alpha_2 & \alpha_2^2\\ \alpha_3^0 & \alpha_3 & \alpha_3^2\end{vmatrix}\begin{matrix}p_3\\p_1\\p_2\end{matrix}=d'$$

which signifies that D has been multiplied by $\alpha_1^{p_3}\alpha_2^{p_1}\alpha_3^{p_2}$ of $\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}$, the first row by $\alpha_1^{p_3}$, the second by $\alpha_2^{p_1}$, and the third by $\alpha_3^{p_2}$, and here as the exponents come to be applied to the rows as the result of the multiplication, they are legitimately applied. Again write D changing columns into rows, and obtain

$$\begin{vmatrix} \alpha_1^0 & \alpha_2^0 & \alpha_3^0 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{vmatrix}$$

Next inquire by what substitution $\alpha_1^{p_3+1}\alpha_2^{p_1+2}\alpha_3^{p_2}$ came from the previous determinant d'. We see that it corresponds to 1st line 2d column, 2d line 3d column, 3d line 1st column, or to the substitution $\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$. Now in order that we may arbitrarily affix the exponents to the rows of D as last written, which is the same as affixing them to the columns as before written, and obtain this term, at least numerically, we must have p_3 with the second line, p_1 with the third, and p_2 with the first. Thus we must have

$$\left|\begin{matrix} \alpha_1^0 & \alpha_2^0 & \alpha_3^0 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{matrix}\right| \begin{matrix} p_2 \\ p_3 \\ p_1 \end{matrix} = d_1$$

which signifies that the exponents p are to be arbitrarily affixed to those of the corresponding line.

What substitution now gives the term in this determinant? We see that it corresponds to 2d line 1st column, 3d line, 2d column, 1st line 3d column, or to $\begin{pmatrix}2 & 3 & 1\\ 1 & 2 & 3\end{pmatrix}$; and this last substitution is the reciprocal of the preceding $\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$ and has the same sign with it. We see farther that to every term of d' corresponds one and only one determinant like d_1, and in d_1 there is one and only one term equal to the given term of d'. To the six terms of d' correspond the six determinants of d_1, d_2, ...d_6; to the six terms of d'' correspond the same six determinants d_1, ...d_6, and so on; to the 36 terms of the six determinants d', d'', ... d^{vi}, correspond the six determints d_1, ...d_6, and conversely to the 36 terms of the six determinants d_1, ...d_6, correspond the six determinants d', d'', ...d^{vi}. Thus the 36 terms of the one set are equal to the 36 terms of the other set, and the results of the two operations are identical.

f. We may express the result of this example more briefly if we write:

$$\left|\begin{matrix} \alpha_1^0 & \alpha_1 & \alpha_1^2 \\ \alpha_2^0 & \alpha_2 & \alpha_2^2 \\ \alpha_3^0 & \alpha_3 & \alpha_3^2 \end{matrix}\right| \begin{matrix} p_{i_1} \\ p_{i_2} \\ p_{i_3} \end{matrix}$$

and take any one of the 36 terms obtained by applying the 3! substitutions $\begin{pmatrix}1 & 2 & 3\\ j_1 & j_2 & j_3\end{pmatrix}$ where the j's are the numbers of the columns, to the six determinants that arise from the 3! permutations of the i's which form a permutation of the numbers 1, 2, 3. We obtain a term of the form

$$\alpha_1^{p_{i_1}+j_1-1}\alpha_2^{p_{i_2}+j_2-1}\alpha_3^{p_{i_3}+j_3-1}.$$

[To be continued.]